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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

Monday 01-02-2010, 09.00-12.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 17 parts. The 18 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

The Hamiltonian corresponding to the Dirac equation in an external electromagnetic field is

$$H = \gamma^0\vec{\gamma}\cdot\vec{\pi} + eA_0(x) + m\gamma^0\,, \tag{1.1}$$

where $\pi_k = p_k + eA_k(x) = -i\big(\partial_k + ieA_k(x)\big)$. We define the **helicity** as the component of the spin of the electron in the direction of the momentum

$$\Sigma = S^k\pi_k\,, \tag{1.2}$$

where the spin $\vec{S} = \frac{1}{2}\gamma^5\gamma^0\vec{\gamma}$.

In the first two parts of this problem we set $e = 0$, i.e., we switch off the electromagnetic field.

1.1 Show that $[\gamma^0, \Sigma] = 0$.

1.2 Show that the helicity is conserved: $[H, \Sigma] = 0$.

Now we set $e \neq 0$ and switch on the electromagnetic field, with vector potential

$$A_0(x) = 0\,, \quad A_k(x) = \tfrac{1}{2}\epsilon_{klm}x^lB^m\,, \tag{1.3}$$

where $\vec{B}$ is a constant vector.

1.3 Show that the electric field corresponding to this vector potential vanishes, and that the magnetic field is given by $\vec{B}$.

1.4 Show that the helicity (which now also contains A_k, see eq. (1.2)!) is still conserved.

PROBLEM 2

A spinor field transforms under Lorentz transformations as

$$\psi'(x') = S(\Lambda)\psi(x)\,, \tag{2.1}$$

where Λ is the Lorentz transformation matrix and $x^{\mu\prime} = \Lambda^{\mu}{}_{\nu}x^{\nu}$.

2.1 Write down the Dirac equation for the free spinor field.

2.2 Prove that covariance of the Dirac equation under Lorentz transformations implies

$$S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}\,. \tag{2.2}$$

2.3 Consider a Lorentz transformation corresponding to a boost with velocity v in the x^1-direction, that is

$$x^{0\prime} = \gamma(x^0 - vx^1)\,, \quad x^{1\prime} = \gamma(-vx^0 + x^1)\,, \quad x^{2\prime} = x^2\,, \quad x^{3\prime} = x^3\,, \tag{2.3}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}\,.$$

Write the Lorentz matrix $\Lambda^{\mu}{}_{\nu}$ for this transformation.

2.4 For infinitesimal Lorentz transformations we write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu}\,. \tag{2.4}$$

Determine $\epsilon^{\mu}{}_{\nu}$ for the transformation in (2.3), with v infinitesimal.

2.5 For infinitesimal Lorentz transformations we set

$$S = \mathbb{1} + \delta S\,.$$

Obtain the relation between δS and $\epsilon^{\mu}{}_{\nu}$ from equations (2.2) and (2.4).

PROBLEM 3

Consider the theory of a scalar field $\phi(x)$, with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_{\mu}\phi(x)\partial^{\mu}\phi(x) - \tfrac{1}{2}m^2\phi(x)\phi(x).$$

3.1 What is the canonical momentum $\pi(x)$ corresponding to the coordinate $\phi(x)$?

3.2 Given that classically $\{\phi(t,\vec{x}), \pi(t,\vec{y})\}_{\mathrm{PB}} = \delta^3(\vec{x}-\vec{y})$, what is the result of the equal-time commutation relation

$$[\phi(x), \pi(y)]_{x^0=y^0} \tag{3.2}$$

for the quantum operators ϕ and π?

3.3 The operator $\phi(x)$ can be written in the form

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left(a(k)e^{-ikx} + a^{\dagger}(k)e^{ikx}\right)_{k^0=\omega_k}\,.$$

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Show that ($x^0 \neq y^0$!)

$$[\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-ik(x-y)} - e^{+ik(x-y)}\right)_{k^0=\omega_k}. \qquad (3.3)$$

3.4 Using (3.3) evaluate

$$[\phi(x), \partial_{y^0}\phi(y)],$$

and show that in the limit $x^0 \to y^0$ the result of (3.2) is obtained.

PROBLEM 4

Consider the annihilation of an electron-positron pair into two photons:

$$e^+ + e^- \to \gamma + \gamma,$$

with momenta

$$e^+ : \ p_1 = (E_1, \vec{p}_1), \ e^- : \ p_2 = (E_2, \vec{p}_2), \ \text{photons}: \ k_1 = (\omega_1, \vec{k}_1), \ k_2 = (\omega_2, \vec{k}_2).$$

The process takes place in the laboratory frame:

$$\vec{p}_2 = 0\,.$$

4.1 Why is $\omega_i = |\vec{k}_i|$?

4.2 Express E_1 and $\vec{p}_1$ in terms of the energies and momenta of the two photons.

4.3 The two photons appear under an angle θ in this process: $\vec{k}_1 \cdot \vec{k}_2 = \omega_1\omega_2\cos\theta$. Express $\cos\theta$ in terms of m, ω_1, ω_2.

4.4 For which energies E_1 do we get $\cos\theta \to 1$?

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